MIT 18.211: COMBINATORIAL ANALYSIS

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Lecture 13: Permutations III

In this lecture, we introduce a new statistic of permutations, the number of descents, and we use this statistic to define Eulerian numbers. We will provide two identity involving the Eulerian numbers.

We say that $i \in [n-1]$ is a *descent* (resp., an *ascent*) of $w = w_1 w_2 \dots w_n \in S_n$ if $w_i > w_{i+1}$ (resp., $w_i < w_{i+1}$). In addition, we set

$$D(w) := \{i \in [n-1] \mid w_i > w_{i+1}\} \quad \text{and} \quad \text{des}(w) := |D(w)|.$$

Definition 1. For $n \in \mathbb{N}$, we let $A(n,k)$ denote the number of permutations in S_n having exactly $k-1$ descents, and we call $A(n,k)$ an *Eulerian number*.

For $n \in \mathbb{N}$, observe that $A(n,k) \neq 0$ only if $k \in [n]$. By convention, we set $A(0,0) = 1$ and $A(0,k) = 0$ if $k \neq 0$.

Example 2. In the extremal cases, we see that $A(n,1) = 1$ as the only permutation of S_n with no descents is the identity permutation, and $A(n,n) = 1$ as the only permutation of S_n with $n-1$ descents is $n(n-1)\dots 1$.

As for Stirling numbers, we can obtain a convenient recurrence identity for Eulerian numbers.

Proposition 3. *For all $n \in \mathbb{N}$ and $k \in \mathbb{N}$, the following recurrence identity holds:*

$$A(n,k) = (n-k+1)A(n-1,k-1) + kA(n-1,k). \tag{0.1}$$

Proof. By definition of the Eulerian numbers, the left-hand side of (0.1) counts the set of permutations in S_n having $k-1$ descents. Set $w' = w_1 w_2 \cdots w_{n-1} \in S_{n-1}$, and suppose we want to insert n in one of the n blanks of the linear arrangement

$$\_\, w_1 \,\_\, w_2 \,\_\, \dots \,\_\, w_{n-2} \,\_\, w_{n-1} \,\_ \tag{0.2}$$

to obtain a permutation $w \in S_n$. If we insert n right after w_i for some $i \in [n-2]$, then $\text{des}(w) = \text{des}(w')$ if and only if $i \in D(w')$ and, otherwise, $\text{des}(w) = \text{des}(w') + 1$. In addition, if we insert n in the last (resp., first) blank of (0.2), then $\text{des}(w) = \text{des}(w')$ (resp., $\text{des}(w) = \text{des}(w') + 1$). Therefore we can also count the set of permutations in S_n with $k-1$ descent as follows. Choose the permutation w' in S_{n-1} among those having $k-1$ descents in $A(n-1,k)$ ways and then insert n in (0.2) either in the last blank or in the blank right after w_i for some descent $i \in D(w')$; this can be done in $kA(n-1,k)$

ways. Now count the rest of the permutations in S_n with $k-1$ descents by choosing a permutation in S_{n-1} with $k-2$ descents and inserting n in (0.2) either in the first blank or after w_i for some ascent position $i \in [n-2] \setminus D(w')$; since the number of ascents of w' is $(n-2)-(k-2)$, this can be done in $(n-k+1)A(n-1,k-1)$. Hence the number of permutations of S_n with $k-1$ descents is $(n-k+1)A(n-1,k-1) + kA(n-1,k)$, which is the right-hand side of (0.1). $\square$

Now we can establish the following polynomial identity for the Eulerian numbers.

Theorem 4. *For each $n \in \mathbb{N}$, the following polynomial identity holds:*

$$x^n = \sum_{k=1}^{n} A(n,k)\binom{x+n-k}{n}. \tag{0.3}$$

Proof. First assume that $x \in \mathbb{N}$. We proceed by induction on n. If $n = 1$, then the left-hand side of (0.3) is x. On the other hand, the right-hand side of (0.3) is

$$A(1,1)\binom{x}{1} = x.$$

Now suppose that (0.3) holds for $n-1 \in \mathbb{N}$. After setting $A_n = \sum_{k=0}^{n} A(n,k)\binom{x+n-k}{n}$, we obtain

$$\begin{aligned} A_n &= \sum_{k=1}^{n}(n-k+1)A(n-1,k-1)\binom{x+n-k}{n} + \sum_{k=1}^{n} kA(n-1,k)\binom{x+n-k}{n} \\ &= \sum_{k=1}^{n-1}(n-k)A(n-1,k)\binom{x+n-k-1}{n} + \sum_{k=1}^{n-1} kA(n-1,k)\binom{x+n-k}{n} \\ &= \sum_{k=1}^{n-1} A(n-1,k)\left[(n-k)\binom{x+n-k-1}{n} + k\binom{x+n-k}{n}\right] \\ &= \sum_{k=1}^{n-1} A(n-1,k)\left[x\binom{(n-1)+x-k}{n-1}\right] = x^n, \end{aligned}$$

where the first equality follows from the recurrence identity in Proposition 3 and the last equality follows from our induction hypothesis. As a result, we have established the identity (0.3) provided that x is a positive integer. Therefore both sides of the same identity are polynomials having infinitely many common roots, and this guarantees that such polynomials are equal. $\square$

Practice Exercises

Exercise 1. *How many permutations in S_n has exactly one descent?*

Exercise 2. *For $n \in \mathbb{N}$ and $k \in [n]$, prove that*

$$A(n,k) = \sum_{j=1}^{k} (-1)^j \binom{n+1}{j} (k-j)^n.$$

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